

AMENABILITY, FØLNER RATIOS, AND COOLING FUNCTIONS

J. W. CANNON, W. J. FLOYD, AND W. R. PARRY

ABSTRACT. Erling Følner proved that the amenability or nonamenability of a countable group G depends on the complexity of its finite subsets $S \subset G$. Complexity has three measures: maximum Følner ratio, optimal cooling function, and minimum cooling norm.

Our first aim is to show that, for a fixed finite subset $S \subset G$, these three measures are tightly bound to one another. We then explore their algorithmic calculation.

Our intent is to provide a theoretical background for algorithmically exploring the amenability and nonamenability of discrete groups.

1. INTRODUCTION

A countable group G is *amenable* if it admits a finitely additive, translation invariant measure defined on every subset of G such that the measure of G is 1. Such groups are important in measure theory, dynamical systems, and ergodic theory. Nonamenable groups lead to the Hausdorff-Banach-Tarski paradox which asserts that *one* copy of the unit ball in \mathbb{R}^3 can be rigidly torn into finitely many pieces that can be rigidly reassembled to form *two* copies of the unit ball.

In [Fol54] and [Fol55], Erling Følner gave two geometric characterizations of countable amenable groups G . These characterizations make these groups very attractive to geometric group theorists. Since we emphasize finite subsets S of G , we shall state the Følner theorems in a slightly nonstandard way. In particular, Følner's statement of Theorem 1.2 is in terms of global cooling functions and not local cooling functions. First we give some introductory definitions.

Let $\Gamma = \Gamma(G, C)$ denote the Cayley graph of an infinite group G with finite generating set C , let $S \subset G$ be nonempty and finite, let $E(S)$ denote the set

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of edges of Γ with at least one vertex in S , and let $\partial E(S)$ denote the set of boundary edges of S consisting of those edges that have exactly one vertex in S . Orient each edge e of $E(S)$ with one of its two possible orientations so that e has an initial vertex $i(e)$ and a terminal vertex $t(e)$. For the edges e of $\partial E(S)$, choose the orientation so that $i(e) \in S$.

The *Følner ratio* of S is the quotient $|S|/|\partial E(S)|$. (We will see that $\partial E(S) \neq \emptyset$.) We will never reduce this fraction except to compare sizes. That is, if we write that the Følner ratio of S is a/b , then we always assume that $a = |S|$ and that $b = |\partial E(S)|$.

A *cooling function* for S is a function $c : E(S) \rightarrow \mathbb{R}$ such that, $\forall s \in S$,

$$h(s) \equiv \sum_{i(e)=s} c(e) - \sum_{t(e)=s} c(e) \geq 1.$$

One can interpret $c(e)$ as the heat pumped along e from $i(e)$ to $t(e)$. Then $h(s)$ is the net loss of heat at s . The *cooling norm* of c is $|c| = \max_{e \in E(S)} |c(e)|$.

Here are the two Følner characterizations. For the purposes of this paper, the reader can take either of these Følner theorems as defining amenability.

Theorem 1.1 (Følner set). *The finitely generated group G is amenable if and only if there exist finite subsets $S_1 \subset S_2 \subset S_3 \subset \cdots$ exhausting G whose Følner ratios $|S_i|/|\partial E(S_i)|$ approach ∞ .*

Theorem 1.2. *The finitely generated group G is nonamenable if and only if there exist finite subsets $S_1 \subset S_2 \subset S_3 \subset \cdots$ exhausting G and admitting cooling functions $c_i : E(S_i) \rightarrow \mathbb{R}$ whose cooling norms $|c_i|$ are uniformly bounded.*

Though no direct use of measure theory appears in this paper, the reader may appreciate an orientation to Følner's procedures. Følner's proofs proceed roughly as follows.

Følner first proves his cooling theorem.

If uniformly bounded cooling functions do not exist, then Følner generalizes an argument of Banach which describes an outer measure on the group and applies a version of the Hahn-Banach theorem to create the desired measure on the group. It is historically interesting to note that Banach's purpose in proving the Hahn-Banach theorem was to prove that Abelian groups are amenable.

If uniformly bounded cooling functions do exist, then the Banach argument fails; and it is fairly easy to show that a measure as in the definition of amenable group cannot possibly exist.

The Følner set theorem then has an easy half and a hard half.

The easy half assumes the existence of subsets whose Følner ratios approach ∞ . Since these sets have relatively small boundaries, it is impossible to pump a lot of heat out of these sets without using cooling functions of large norm. Hence cooling functions of uniformly bounded norm do not exist and the group is amenable.

The hard half assumes that such sets do not exist and, by an extremely indirect proof, Følner proves that cooling functions with uniformly bounded norms do exist.

It is the hard half of this proof that has most interest for us. Our goal is to make Følner's indirect proof as direct and as algorithmic as possible. In the process, we streamline the Følner proof.

Statement 2 of Lemma 2.1 shows that cooling functions for S always exist in this context. Using this, it is easy to show that the two theorems are equivalent by application of our main theorem:

Theorem 1.3. *Let Γ be a locally finite graph with vertex set G and a nonempty finite subset $S \subset G$ which admits a cooling function. Then S admits a cooling function c of minimum possible cooling norm $N = |c|$, and*

$$N = |c| = \max_{S_0 \subset S} |S_0|/|\partial E(S_0)|.$$

That is, the minimum cooling norm is equal to the maximum Følner ratio of subsets S_0 of S , taking the Følner ratio of the empty set to be 0. We call such a cooling function with minimum cooling norm an *optimal cooling function*. This absolute cooling theorem will be a corollary to the cooling theorem of Section 2.

We prove the cooling theorem in Section 2 by refining Følner's original arguments. In Section 3 we interpret this cooling theorem in terms of linear programming and give another proof using standard results of linear programming. Section 3 concludes with a brief derivation of statement 2 of the cooling theorem using the max flow min cut theorem. The linear programming approach in Section 3 leads to an algorithm, presented in Section 4, which is a modification of the standard simplex algorithm. It finds an optimal cooling function c and a Følner-optimal subset $S_0 \subset S$, that is, a subset

$S_0 \subset S$ whose Følner ratio is the maximum possible. Moreover, the subset S_0 which the algorithm finds has the property that if S'_0 is a Følner-optimal subset of S , then $S'_0 \subset S_0$. In other words, there is a maximal such subset of S with maximum Følner ratio, and the algorithm finds it.

After Section 4, we describe our partial results toward another algorithmic process for finding the minimum possible cooling norm N , finding an optimal cooling function, and finding a Følner-optimal subset $S_0 \subset S$. The main ingredient is the relative cooling theorem, another corollary to the cooling theorem of Section 2 (see Section 5). Then we describe the process of peeling layers away from an arbitrary set to find a subset of maximum possible Følner ratio (Section 6). And finally we show how to build up optimal cooling functions layer by layer from known cooling functions on subsets (Section 7).

It can be challenging to determine whether or not a countable group is amenable. In particular, despite concerted efforts for over 30 years it is still not known whether or not Thompson's group F is amenable. And while Bartholdi and Virág showed in [BV05] that the Basilica group is amenable, their proof is via random walks and one still doesn't know how to construct a Følner sequence for it. Our hope is that the results of this paper will serve in exploring amenability for countable groups.

Permanent setting for the remainder of the paper: Γ is a locally finite graph; G is the set of vertices of Γ ; S is a nonempty finite subset of G ; $E(S)$ is the set of edges of Γ having at least one vertex in S , and $\partial E(S) \subset E(S)$ is the set of edges of Γ having exactly one vertex in S .

Always we assume that the set S admits a cooling function.

Edge orientation: For each edge $e \in E(S)$, we choose one of the two possible orientations of e , so that e has an *initial vertex* $i(e)$ and a *terminal vertex* $t(e)$. If $e \in \partial E(S)$, then we choose that orientation which has $i(e) \in S$ and $t(e) \notin S$.

We always assume given, as initial data, a function $h_0 : S \rightarrow (0, \infty)$. In most applications, this function will be identically equal to 1, but in the relative cooling theorem it is important that we be allowed to modify this initial data.

2. THE COOLING THEOREM

Cooling functions: A *cooling function* *rel* h_0 is a function $c : E(S) \rightarrow \mathbb{R}$ such that

$$\forall s \in S, \quad h(s) \equiv \sum_{i(e)=s} c(e) - \sum_{t(e)=s} c(e) \geq h_0(s).$$

We may interpret $c(e)$ as the heat pumped along edge e from initial vertex $i(e)$ to terminal vertex $t(e)$. The function $h(s)$ is then the net heat loss at the vertex s . The absolute cooling theorem of the introduction deals with the special case where h_0 is constant and equal to 1. The value $h_0(s)$ gives a lower bound on the amount of heat to be pumped out of vertex s . The sum $H(S) = \sum_{s \in S} h_0(s)$ gives the minimal amount of heat that a cooling function must pump out of S through the boundary edges $\partial E(S)$ of S .

Recall that we always assume that S admits a cooling function. The next lemma provides an equivalent condition. We need two definitions for this. Let T be a finite subset of G . A connected component of T consists of all the vertices in a connected component of the graph $\cup\{e : e \in E(T) \setminus \partial E(T)\}$. To say that a connected component T_0 of T has a nonempty boundary, we mean that $\partial E(T_0) \neq \emptyset$.

Lemma 2.1. (1) *A nonempty finite subset T of G admits a cooling function *rel* h_0 if and only if every connected component of T has a nonempty boundary.*

(2) *If Γ is infinite and connected, then every nonempty finite subset $T \subset G$ admits a cooling function *rel* h_0 .*

Proof. Let T be a nonempty finite subset of G .

We first prove the forward implication of statement 1. Suppose that T admits a cooling function *rel* h_0 . Let T_0 be a connected component of T . Then $H(T_0) > 0$. Edges of $E(T_0) \setminus \partial E(T_0)$ conduct no heat from T_0 . Thus $\partial E(T_0) \neq \emptyset$. This proves the forward implication of statement 1.

For the backward implication, suppose that every connected component of T has a nonempty boundary. Then for each $x \in T$ there exists an edge path $e_1 = (x_0, x_1)$, $e_2 = (x_1, x_2)$, \dots , $e_n = (x_{n-1}, x_n)$ with $x = x_0$, with $x_i \in T$ for $i < n$, and with $e_n \in \partial E(T)$, so that $x_n \notin T$. Given x , transport $h_0(x)$ units of heat from $x = x_0$ to x_1 , the same from x_1 to x_2 , etc., until $h_0(x)$ units have been, in effect, transported from x_0 to x_n with no net gain at any intermediate vertex. That is, viewing e_i as oriented from x_{i-1} to x_i , we define a function from $E(T)$ to \mathbb{R} so that its value at e_1, \dots, e_n is $h_0(x)$ and

0 otherwise. Call this function a heat-flow path. Adding one heat-flow path for each $x \in T$ results in a function $c : E(T) \rightarrow \mathbb{R}$ satisfying $h(x) = h_0(x)$ for each $x \in T$. Thus c is a cooling function rel h_0 .

This proves statement 1 of Lemma 2.1.

To prove statement 2, suppose that Γ is infinite and connected. Let $x \in T$. Because Γ is infinite and T is finite, there exists $y \in G \setminus T$. Because Γ is connected, there exists an edge path in Γ joining x and y . Some edge in this edge path is in $\partial E(T)$. It follows that every connected component of the boundary of T is nonempty. Statement 2 now follows from statement 1.

This proves Lemma 2.1. □

Følner ratio: Because we always assume that S admits a cooling function rel h_0 , Lemma 2.1 implies that $\partial E(S) \neq \emptyset$. The *Følner ratio* $FR(S) = H(S)/|\partial E(S)|$ measures the average amount of heat that must be pumped out of S along each edge $e \in \partial E(S)$ by any cooling function rel h_0 . Since S admits a cooling function rel h_0 , so does every nonempty subset S_0 of S . So $FR(S_0)$ exists for each S_0 . We set $FR(\emptyset) = 0$.

Lemma 2.2. *If $S_0 \subset S$, then $FR(S_0)$ is a lower bound on the norm of every cooling function on S rel h_0 .*

Proof. Since a cooling function must cool each subset $S_0 \subset S$, since the heat in S_0 must be carried out of S_0 along the boundary edges of S_0 , and since at least one boundary edge must carry at least the average required per boundary edge, it follows immediately that the minimum cooling norm $N = |c|$ of a cooling function must be at least as large as the Følner ratio $FR(S_0)$. □

Our absolute cooling theorem will show that N is exactly equal to the maximum of the Følner ratios which occur in this lemma.

The simplex $\Delta(S)$ with vertices S and the functions \tilde{f} : Let $\Delta(S)$ denote the abstract simplex whose vertices are the elements of the set S . An element of $\Delta(S)$ is therefore a function $f : S \rightarrow [0, 1]$ such that $\sum_{s \in S} f(s) = 1$. With each element $f \in \Delta(S)$ we associate a function $\tilde{f} : V \rightarrow [0, \infty)$, where

$$\begin{aligned} V &= \cup_{e \in E(S)} \partial e, \\ \forall v \in S, \quad \tilde{f}(v) &= f(v)/h_0(v), \text{ and} \\ \forall v \in V \setminus S, \quad \tilde{f}(v) &= 0. \end{aligned}$$

Theorem 2.3 (Cooling). (1) If N is the minimum possible norm of a cooling function for S rel h_0 , then the reciprocal $1/N$ is given by the minimum value of a convex function $A(f)$ on the simplex $\Delta(S)$ as follows:

$$1/N = \min_{f \in \Delta(S)} A(f),$$

where \tilde{f} is the modification of f defined above and

$$A(f) = \sum_{e \in E(S)} |\tilde{f}(i(e)) - \tilde{f}(t(e))|.$$

(2) The minimum possible norm N is given by the maximum Følner ratio:

$$N = \max_{S_0 \subset S} FR(S_0).$$

(3) If $FR(S_0)$ realizes the maximum in (2), then the function $f \in \Delta(S)$ whose modification \tilde{f} is constant on S_0 and 0 on $V \setminus S_0$ realizes the minimum in (1).

Proof. (1): The verification that A is a convex function is left to the reader. Since the function $A : \Delta(S) \rightarrow (0, \infty)$ is positive and continuous on the compact set $\Delta(S)$, there is certainly a function in $\Delta(S)$ which gives a positive minimum for A .

For the rest, we follow Følner's argument. We fix a candidate norm N for a cooling function rel h_0 . We introduce a variable y_e for each $e \in E(S)$ and note that a cooling function of norm $\leq N$ is an assignment of a real value to each of the variables y_e in such a way that the following inequalities are satisfied:

$$\forall s \in S, \quad h(s) = \sum_{i(e)=s} y_e - \sum_{t(e)=s} y_e \geq h_0(s); \text{ and}$$

$$\forall e \in E(S), \quad y_e \geq -N \quad \text{and} \quad -y_e \geq -N.$$

The first of these inequalities says that each element of S is cooled by the required amount. The last two inequalities say that the norm of the resulting cooling function is $\leq N$.

We assume that N is too small, so that no cooling function exists, and conclude from Farkas' lemma, Lemma 2.9, that there exist nonnegative numbers $\alpha(s)$, $\beta(e)$, and $\gamma(e)$ satisfying the following two conditions:

$$(2.4) \quad \sum_{s \in S} \alpha(s) h(s) + \sum_{e \in E(S)} (\beta(e) - \gamma(e)) y_e \equiv 0,$$

and

$$(2.5) \quad \sum_{s \in S} \alpha(s) h_0(s) - N \cdot \sum_{e \in E(S)} (\beta(e) + \gamma(e)) > 0.$$

We cannot have $\alpha(s)$ identically 0, for then the left-hand side of line 2.5 would be ≤ 0 , a contradiction. We may therefore scale the coefficients $\alpha(s)$, $\beta(e)$, and $\gamma(e)$ so that $\sum \alpha(s) h_0(s) = 1$. That is, we may assume that $f = \alpha \cdot h_0 \in \Delta(S)$. We may assume the function α is extended to the vertices of $\partial E(S)$ not in S so as to be 0 on vertices not in S . It follows that this extended α is our standard modification \tilde{f} of the function f .

We normalize $\beta(e)$ and $\gamma(e)$ as follows: We replace the larger of $\beta(e)$ and $\gamma(e)$ by the nonnegative difference $|\beta(e) - \gamma(e)|$ and the other by 0. This has no effect at all on the condition in line 2.4, and it can only increase the sum in line 2.5; so the two conditions are still satisfied.

We claim that $\beta(e) + \gamma(e) = |\alpha(i(e)) - \alpha(t(e))| = |\tilde{f}(i(e)) - \tilde{f}(t(e))|$. Indeed, the coefficient of y_e in line 2.4 is 0, and so

$$(2.6) \quad \alpha(i(e)) - \alpha(t(e)) + \beta(e) - \gamma(e) = 0.$$

Because of our normalization, one of $\beta(e)$ and $\gamma(e)$ is 0, and so $|\beta(e) + \gamma(e)| = |\beta(e) - \gamma(e)|$. Hence

$$\beta(e) + \gamma(e) = |\beta(e) + \gamma(e)| = |\beta(e) - \gamma(e)| = |\alpha(i(e)) - \alpha(t(e))|,$$

as desired.

With $\sum \alpha(s) h_0(s) = 1$ and $\beta(e) + \gamma(e) = |\alpha(i(e)) - \alpha(t(e))| = |\tilde{f}(i(e)) - \tilde{f}(t(e))|$, we find that line 2.5 is equivalent to

$$A(f) = \sum_{e \in E(S)} |\tilde{f}(i(e)) - \tilde{f}(t(e))| < 1/N.$$

Thus cooling functions of norm N exist if $1/N \leq \min_{f \in \Delta(S)} A(f)$.

The converse of this statement can be proved by assuming that there exists $f \in \Delta(S)$ with $A(f) < 1/N$ and reversing this argument. The function f determines the numbers $\alpha(s)$ as above. We choose the numbers $\beta(e)$ and $\gamma(e)$ as above so that one of them is 0 and so that line 2.6 is satisfied. We conclude that lines 2.4 and 2.5 hold. Now Farkas' lemma implies that there is no cooling function with norm N . So cooling functions with norm N exist if and only if $1/N \leq \min_{f \in \Delta(S)} A(f)$.

This proves (1).

(2) and (3): Lemma 2.2 shows that the minimum possible cooling norm N for S must be at least as large as $\max_{S_0 \subset S} FR(S_0)$. Our remaining task is

to find a subset S_0 of maximum Følner ratio and to show that the function $g \in \Delta(S)$ associated with S_0 as in (3) does in fact realize the minimum $A(g) = \min_{f \in \Delta(S)} A(f)$ in (1).

Suppose that A assumes its minimum at $f \in \Delta(S)$. We let $S_0 \subset S$ denote the set of points in S at which the corresponding function \tilde{f} takes its (positive) maximum. We shall see that this set S_0 satisfies the required conditions.

As in (3), we let \tilde{g} be constant and positive, equal to δ , on S_0 , and constant, equal to 0, on the complement of S_0 . The corresponding element $g \in \Delta(S)$ therefore satisfies the equation

$$1 = \sum_{s \in S} g(s) = \sum_{s \in S_0} \delta \cdot h_0(s) = \delta \cdot H(S_0).$$

Thus $\delta = 1/H(S_0)$. It is easy to calculate $A(g)$ since $|\tilde{g}(i(e)) - \tilde{g}(t(e))|$ is only nonzero for $e \in \partial E(S_0)$. Therefore,

$$(2.7) \quad A(g) = \sum_{e \in E(S)} |\tilde{g}(i(e)) - \tilde{g}(t(e))| = \delta \cdot |\partial E(S_0)| = \frac{|\partial E(S_0)|}{H(S_0)} = \frac{1}{FR(S_0)}.$$

We know that $A(f)$ realizes the minimum in (1). We complete the proof by showing that $A(f) \geq A(g)$ so that the latter also realizes the minimum in (1).

We apply the methods of the calculus of variations. We consider the functions $f_\lambda \in \Delta(S)$ defined as follows:

$$f_\lambda = \frac{f - \lambda g}{1 - \lambda}, \text{ for } \lambda \text{ very small in } [0, 1).$$

Note that we are simply reducing some of the positive values of f by a little bit and are then scaling so as to remain in $\Delta(S)$. The following lemma completes the proof.

Lemma 2.8.

$$0 \leq \left. \frac{d(A(f_\lambda))}{d\lambda} \right|_{\lambda=0} = A(f) - A(g).$$

Proof. The derivative is certainly ≥ 0 since $A(f)$ minimizes A .

It is an easy matter to take the derivative of $|x|$ if we know which of the two options $|x| = 1 \cdot x$ or $|x| = -1 \cdot x$ is true. Luckily, we can determine appropriate signs ± 1 for each of our three absolute values $|\tilde{f}(i(e)) - \tilde{f}(t(e))|$, $|\tilde{f}_\lambda(i(e)) - \tilde{f}_\lambda(t(e))|$, and $|\tilde{g}(i(e)) - \tilde{g}(t(e))|$, which appear in the defining formula for $A(f)$, $A(f_\lambda)$, and $A(g)$, and those signs may be chosen compatibly.

To each edge $e \in E(S)$ we assign a number $\epsilon(e) = \pm 1$ as follows: if $|\tilde{f}(i(e)) - \tilde{f}(t(e))| = \tilde{f}(i(e)) - \tilde{f}(t(e))$, then $\epsilon(e) = 1$; otherwise, $\epsilon(e) = -1$.

For λ sufficiently small, the function \tilde{f}_λ will clearly require the same signs $\epsilon(e)$.

Because the function \tilde{g} is positive only on the elements of S_0 , where \tilde{f} takes on its maximum value, \tilde{g} will require the same sign provided that exactly one end point of e is in S_0 . If neither end point of e is in S_0 or if both end points of e are in S_0 , then $\tilde{g}(i(e)) - \tilde{g}(t(e)) = 0$ and we may use the same value of $\epsilon(e)$ for \tilde{g} as has already been chosen for \tilde{f} and \tilde{f}_λ .

Thus we may choose multipliers $\epsilon(e) = \pm 1$ so that we have the following three equalities for all λ sufficiently small:

$$\begin{aligned} |\tilde{f}(i(e)) - \tilde{f}(t(e))| &= \epsilon(e) \cdot (\tilde{f}(i(e)) - \tilde{f}(t(e))) \\ |\tilde{f}_\lambda(i(e)) - \tilde{f}_\lambda(t(e))| &= \epsilon(e) \cdot (\tilde{f}_\lambda(i(e)) - \tilde{f}_\lambda(t(e))) \\ |\tilde{g}(i(e)) - \tilde{g}(t(e))| &= \epsilon(e) \cdot (\tilde{g}(i(e)) - \tilde{g}(t(e))). \end{aligned}$$

We would be unable to obtain compatible signs $\epsilon(e)$ if S_0 had been chosen via anything but the *maximum* value of \tilde{f} . With all absolute value signs replaced by constants $\epsilon(e)$, it is an easy matter to calculate $d(A(f_\lambda))/d\lambda$ by the quotient rule, and the result is as stated in the lemma:

$$\begin{aligned} d(A(f_\lambda))/d\lambda \Big|_{\lambda=0} &= \sum_e \epsilon(e) (\tilde{f}(i(e)) - \tilde{f}(t(e))) \frac{d}{d\lambda} \left(\frac{1}{1-\lambda} \right) \Big|_{\lambda=0} \\ &\quad - \sum_e \epsilon(e) (\tilde{g}(i(e)) - \tilde{g}(t(e))) \frac{d}{d\lambda} \left(\frac{\lambda}{1-\lambda} \right) \Big|_{\lambda=0} \\ &= A(f) - A(g). \end{aligned}$$

□

This argument completes the proof of the cooling theorem.

□

Proof of the absolute cooling theorem. Set $h_0 \equiv 1$.

□

We conclude this section with a discussion of Farkas' lemma. In Section 4 of [Fol55], Følner states essentially the following result, which he presents as a minor modification of a result proved by Carver in Theorem 3 of [Car22].

Lemma 2.9 (Farkas' lemma). *Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and let $b \in \mathbb{R}^m$. Then a necessary and sufficient condition that there be no solution*

to the inequality $L(x) \geq b$ is that there exist a nonnegative vector $\alpha \in \mathbb{R}^m$, $\alpha \geq 0$, such that $L(x) \cdot \alpha \equiv 0$ but $b \cdot \alpha > 0$.

This is one of the equivalent formulations of Farkas' lemma. To see this, we apply Corollary 7.1e on page 89 of Schrijver's book [Sch99]. This corollary is one form of Farkas' lemma. It states the following, a bit loosely. Let A be a matrix and let b be a vector. Then the system $Ax \leq b$ of linear inequalities has a solution x , if and only if $yb \geq 0$ for each row vector $y \geq 0$ with $yA = 0$. Replacing b with $-b$, yields the following. The system $Ax \geq b$ of linear inequalities has a solution x , if and only if $yb \leq 0$ for each row vector $y \geq 0$ with $yA = 0$. Lemma 2.9 is the contrapositive of this.

3. LINEAR PROGRAMMING

In this section we give another proof of the cooling theorem, Theorem 2.3, using standard results from linear programming. Murty [Mur76] and Schrijver [Sch99] are good references for linear programming. We maintain the notation of Section 2.

We wish to minimize the maximum value of

$$c(e), -c(e) \quad \forall e \in E(S)$$

subject to the conditions

$$\sum_{i(e)=s} c(e) - \sum_{t(e)=s} c(e) \geq h_0(s) \quad \forall s \in S.$$

In the spirit of linear programming, we introduce a variable x_e for every $e \in E(S)$ and restate this problem as follows. Minimize the maximum value of

$$x_e, -x_e \quad \forall e \in E(S)$$

subject to the conditions

$$\sum_{i(e)=s} x_e - \sum_{t(e)=s} x_e \geq h_0(s) \quad \forall s \in S.$$

Now we introduce one more variable x_h and in effect replace x_e by x_e divided by the maximum value of $\{|x_e| : e \in E(S)\}$ to see that our problem is equivalent to the following problem. Maximize x_h subject to the conditions

$$(3.1) \quad \begin{aligned} h_0(s)x_h - \sum_{i(e)=s} x_e + \sum_{t(e)=s} x_e &\leq 0 \quad \forall s \in S \\ \left. \begin{aligned} x_e &\leq 1 \\ -x_e &\leq 1 \end{aligned} \right\} &\quad \forall e \in E(S). \end{aligned}$$

The solution of this problem is the inverse of the solution of our original problem.

This is a linear programming problem in the form $\max\{cx : Ax \leq b\}$ as in line 13 on page 90 of Schrijver's book [Sch99]. Here x is the column vector of variables x_h, x_e for $e \in E(S)$. We have that c is the row vector with as many components as x with the component corresponding to x_h being 1 and all other components being 0. Similarly, b is the column vector of right side constants in line 3.1 and A is the coefficient matrix of this system of linear inequalities. (The A here is not to be confused with our previous A .)

Now we apply the duality theorem of linear programming. Just as Farkas' lemma, which we encountered at the end of Section 2, has a number of formulations, so does the duality theorem. We apply the formulation given in Corollary 7.1g on page 90 of Schrijver's book [Sch99]:

$$\max\{cx : Ax \leq b\} = \min\{yb : y \geq 0, yA = c\},$$

provided that both of these sets are nonempty. The origin is in the first set, so the first set is nonempty. We will soon see that the second one is nonempty too.

So we now have new variables y_s for every $s \in S$, one for every inequality in line 3.1 corresponding to an element of S . We also have new variables y_e and z_e for every $e \in E(S)$ corresponding to the two inequalities in line 3.1 for every $e \in E(S)$. Furthermore y is the row vector whose entries are these new variables. The duality theorem transforms our problem into the following one. Minimize

$$\sum_{e \in E(S)} (y_e + z_e)$$

subject to the conditions

$$\begin{aligned} \sum_{s \in S} h_0(s)y_s &= 1 \\ -y_{i(e)} + y_{t(e)} + y_e - z_e &= 0 \quad \forall e \in E(S) \\ y_s &\geq 0 \quad \forall s \in S \quad y_e \geq 0 \quad \forall e \in E(S) \quad z_e \geq 0 \quad \forall e \in E(S). \end{aligned}$$

The last equation requires the convention that $y_{t(e)} = 0$ if $t(e) \notin S$.

Suppose that we fix the variables y_s and let the variables y_e and z_e vary. Then $y_e - z_e$ is fixed for every $e \in E(S)$. If y and z are nonnegative real numbers with $y - z = a$, a fixed value, then the smallest possible value for $y + z$ is $|a|$. Thus our problem is equivalent to the following one. Minimize

$$\sum_{e \in E(S)} |y_{i(e)} - y_{t(e)}|$$

subject to the conditions

$$\sum_{s \in S} h_0(s) y_s = 1$$

$$y_s \geq 0 \quad \forall s \in S.$$

Now it is clear that our second set is also nonempty, and we have statement 1 of the cooling theorem.

To prove statements 2 and 3 of the cooling theorem, note that the map A is not only convex but piecewise linear. In fact, its restriction to every cell of the first barycentric subdivision of $\Delta(S)$ is linear. As for general linear programming problems, it follows that its minimum occurs at a vertex of the first barycentric subdivision of $\Delta(S)$. (See Section 3.5.5 of Murty's book [Mur76] or Section 8.3 of Schrijver's book [Sch99].) These vertices are the points $f \in \Delta(S)$ such that f is constant on some subset S_0 of S and 0 on $S \setminus S_0$. From the definition of A , we see that if f is such a point of $\Delta(S)$, then

$$A(f) = \frac{|\partial E(S_0)|}{|S_0|} = \frac{1}{FR(S_0)},$$

as in line 2.7.

This completes our linear programming proof of the cooling theorem.

We conclude this section with a brief discussion involving the max flow min cut theorem. Given the close connection between the max flow min cut theorem and linear programming, (See Section 7.10 of Schrijver's book [Sch99].) it should not be surprising that the max flow min cut theorem is relevant to our problem.

We apply Theorem 1.1 on page 38 of Ford and Fulkerson's book [FF62] We take the sets R and S there to be empty and the set T there to be our set S . The set \overline{X} there is our S_0 . The function f there is our cooling function c . The function b there is our h_0 . The function c there is our norm $n = |c|$. The conclusion is that the real number n is the norm of a cooling function c on S rel h_0 if and only if $H(S_0) \leq n |\partial E(S_0)|$ for every $S_0 \subset S$. Thus $N = \max_{S_0 \subset S} FR(S_0)$. This is statement 2 of the cooling theorem.

4. THE MODIFIED SIMPLEX ALGORITHM

The previous section shows that the problem of finding an optimal cooling function is a linear programming problem. As such, it can be solved by the simplex method. In our situation every step of the simplex method proceeds as follows. We have a cooling function c , and the simplex method finds the

vertices at which c is not optimal. One chooses one of these vertices, and the step concludes with a computation which improves c at the chosen vertex. This algorithm has the defects that at every step the norm of c usually does not decrease, there is this choice of vertex, and poor choices might lead to an infinite loop, although a further enhancement of the algorithm can avoid infinite loops.

The algorithm which we present in this section is a greedy algorithm in that at every step we improve our cooling function c at every vertex at which c is not optimal. Because we deal with every bad vertex, the norm of c decreases at every step. In our algorithm the numerator of $|c|$ is always bounded by $|S|$ and the denominator of $|c|$ is bounded by $|E(S)|$. Since $|c|$ decreases at every step, the number of steps is therefore bounded by $|S| |E(S)|$. The number of operations per step is linear in $|E(S)|$. Thus the number of operations required for our algorithm to find the optimal cooling function is cubic in $|E(S)|$. In practice it seems to be quadratic. The steps in our algorithm are far more complicated than the steps of the simplex method, but there are far fewer of them. In our limited experience, our algorithm is faster. Moreover, the subset S_0 of S with maximal Følner ratio which our algorithm finds also has the property (See Theorem 4.1 and the discussion immediately preceding it.) that if T is a subset of S with maximal Følner ratio, then $T \subset S_0$.

Here is our modified simplex algorithm.

Let Γ be a locally finite graph, as usual. The edges of Γ are initially undirected. However, we often direct the edges of Γ , choosing directions to suit the occasion. If e is a directed edge of Γ , then, as usual, we let $i(e)$ denote the initial vertex of e , and we let $t(e)$ denote the terminal vertex of e .

Now we fix a nonempty finite set S of vertices of Γ , as usual. We direct the edges of $E(S)$ so that every edge of $\partial E(S)$ is directed away from S . Let h be a nonnegative real number. A cooling function for S relative to h is a function $c : E(S) \rightarrow \mathbb{R}$ such that

$$\sum_{i(e)=v} c(e) - \sum_{t(e)=v} c(e) \geq h \quad \text{for every } v \in S.$$

We refer to this inequality as the vertex condition or vertex inequality at v .

The absolute cooling theorem deals with the case in which $h = 1$. It states that S admits a cooling function relative to $h = 1$ of minimum norm, and this norm is $N = \max_{S_0 \subset S} FR(S_0)$. In other words, there exists a maximum value of h such that S admits a cooling function relative to h with norm

1, and this maximum value of h is $N^{-1} = \min_{S_0 \subset S} FR(S_0)^{-1}$. In this way we are led to the problem of maximizing h over all cooling functions for S with norm 1. This is what our modified simplex algorithm does: it maximizes h over all cooling functions for S with norm 1. For every edge $e \in E(S)$, we refer to the inequality $|c(e)| \leq 1$ as the edge condition or edge inequality at e .

Our modified simplex algorithm proceeds in steps, starting with step 0. After n steps we have the following. We have a forest F_n , which is a subgraph of Γ . The vertex set of F_n equals S . Every connected component of F_n is a rooted tree. We direct every edge of F_n down toward the root of its component. We also have a set $R_n \subset E(S) \setminus \partial E(S)$. No element of R_n is an edge of F_n , and every element of R_n is directed. We direct the edges of $E(S)$ compatibly with F_n , R_n and $\partial E(S)$. We also have a nonnegative real number h_n . A cooling function for (S, F_n, R_n, h_n) is a function $c : E(S) \rightarrow \mathbb{R}$ satisfying the following conditions.

- (1) $c(e) = 1$ if $e \in R_n \cup \partial E(S)$
- (2) $c(e) = 0$ if e is neither an edge of F_n nor in $R_n \cup \partial E(S)$
- (3) $\sum_{i(e)=v} c(e) - \sum_{t(e)=v} c(e) = h_n$ for every element v of S which is not a root of F_n
- (4) $\sum_{i(e)=v} c(e) - \sum_{t(e)=v} c(e) \geq h_n$ for every root v of F_n

In addition to the above, we have a cooling function c_n for (S, F_n, R_n, h_n) with norm 1 which never takes the value -1 . Finally, h_n is maximal with respect to the property that there exists a cooling function for (S, F_n, R_n, h_n) with norm 1. Given such a quadruple (F_n, R_n, h_n, c_n) , the algorithm constructs another one $(F_{n+1}, R_{n+1}, h_{n+1}, c_{n+1})$ with $h_{n+1} > h_n$ unless the algorithm finds a subset of S with maximum Følner ratio. This eventually occurs, the algorithm finds a subset of S with maximum Følner ratio h_n^{-1} together with an associated cooling function and the algorithm stops.

The situation at step 0 is as simple as possible. The forest F_0 has no edges. Its connected components, the elements of S , are trivial rooted trees. The set R_0 is empty. The cooling function c_0 takes the value 1 on $\partial E(S)$ and the value 0 elsewhere. The maximality of h_0 then implies that

$$h_0 = \min_{v \in S} |\{e \in \partial E(S) : i(e) = v\}|.$$

All conditions are satisfied.

Now let n be a nonnegative integer, and suppose that we have F_n , R_n , h_n and c_n as above. We prepare to construct F_{n+1} , R_{n+1} , h_{n+1} and c_{n+1} in the next paragraph.

Suppose that the edge inequality satisfied by c_n is strict for every edge of F_n . Let T be a nontrivial connected component of F_n , and let v be a leaf of T . (Roots are not leaves.) Because the edge inequality satisfied by c_n at the edge e of T which contains v is strict, it is possible to send slightly more than $c_n(e)$ units of heat toward the root of T along e while satisfying the vertex equality at v for a value slightly larger than h_n . We modify c_n in this way at every edge of T to satisfy the vertex equality at every vertex of T other than its root for a real number slightly larger than h_n . This applies to every such connected component of F_n . Thus because h_n is maximal, the vertex inequality satisfied by c_n for h_n at some root of F_n is actually an equality. We conclude that either the edge inequality satisfied by c_n at some edge of F_n is actually an equality or the vertex inequality satisfied by c_n for h_n at some root of F_n is actually an equality. We observe that because c_n never takes the value -1 , if e is an edge of F_n , then $|c_n(e)| = 1$ if and only if $c_n(e) = 1$.

We digress briefly in this paragraph to show that h_n^{-1} is a relative Følner ratio. According to the previous paragraph there exists either a root or an edge of F_n such that the inequality satisfied by c_n at either this root or edge is actually an equality. If this equality holds at a root, then let v be this root, and if this equality holds at an edge, then let v be the upper vertex of this edge. Let S_0 be the set of those vertices in S which are either equal to or above v relative to F_n . By assumption the vertex inequality satisfied by c_n for h_n is actually an equality at every element of S_0 other than v , and now this is even true at v . We combine these equations, one for every vertex of S_0 . We use the facts that $c_n(e) = 1$ for every edge $e \in R_n \cup \partial E(S)$ and $c_n(e) = 0$ if e is neither an edge of F_n nor in $R_n \cup \partial E(S)$. We also use the fact that if v is not a root, then the edge of F_n immediately below it is directed away from S_0 and c_n has value 1 at it. Letting $\partial' E(S_0)$ denote the set of edges of either $\partial E(S)$ or F_n which contain exactly one element of S_0 , we obtain that

$$|\partial' E(S_0)| + \sum_{\substack{e \in R_n \\ i(e) \in S_0}} 1 - \sum_{\substack{e \in R_n \\ t(e) \in S_0}} 1 = |S_0| h_n.$$

Hence $h_n = RFR(S_0)^{-1}$, where

$$RFR(S_0) = |S_0| \left(|\partial' E(S_0)| + \sum_{\substack{e \in R_n \\ i(e) \in S_0}} 1 - \sum_{\substack{e \in R_n \\ t(e) \in S_0}} 1 \right)^{-1}.$$

We view $RFR(S_0)$ as a relative Følner ratio. It is the Følner ratio of S_0 relative to S , F_n and R_n .

We return to the result of the penultimate paragraph. Either the edge inequality satisfied by c_n at some edge of F_n is actually an equality or the vertex inequality satisfied by c_n for h_n at some root of F_n is actually an equality. We find every such root and edge. Every such root loses its status as root. Every such edge, but not its vertices, is removed from F_n . The result is a forest, which we denote by F'_n . Every edge just removed from F_n , directed as in F_n , is added to R_n . The result is a set of directed edges, which we denote by R'_n . The forest F'_n has connected components without roots. We now construct F_{n+1} by inductively enlarging the rooted connected components of F'_n . We adjoin certain edges to F'_n to construct a maximal forest with at most one root per connected component. The edges which we adjoin are edges of $E(S) \setminus \partial E(S)$ which are either not in R'_n or directed away from the rooted connected components. So, if possible we adjoin such an edge to F'_n , joining one of its rooted connected components with one of its unrooted connected components. The result is a forest, one of whose rooted connected components contains two connected components of F'_n . If possible we adjoin such an edge to this new forest, joining one of its rooted connected components with one of its unrooted connected components. We continue in this way as long as possible. This final forest is F_{n+1} .

Suppose that F_{n+1} has a connected component which is not rooted (which occurs if F'_n has no roots). Let S_0 be the set of elements of S which are not contained in rooted connected components of F_{n+1} . Then every edge in $\partial E(S_0)$ is either in $\partial E(S)$ or it is in R'_n and directed away from S_0 . Hence $c_n(e) = 1$ for every $e \in \partial E(S_0)$. It follows that c_n is removing as much heat from S_0 as is possible while maintaining norm 1. As in the paragraph which discusses relative Følner ratios, there is no cooling function for S with norm 1 relative to a value larger than $h_n = RFR(S_0)^{-1} = FR(S_0)^{-1}$, where this relative Følner ratio is computed relative to R'_n . In this case S_0 is a subset of S with maximum Følner ratio, and the algorithm stops.

Suppose that the algorithm does not stop at step n . Then every connected component of F_{n+1} is rooted. We define R_{n+1} to be the set of directed edges gotten from R'_n by deleting those edges of R'_n which became edges of F_{n+1} . If v is a root of F_n at which the inequality satisfied by c_n for h_n is an equality, then v is not a root of F_{n+1} . If e is an edge of F_n at which the inequality satisfied by c_n is an equality, that is, $c_n(e) = 1$, then either e is not an edge of F_{n+1} or it is directed away from the root of its connected component. The cooling function c_n need not be a cooling function for $(S, F_{n+1}, R_{n+1}, h_n)$ mostly because some edge directions might have changed. By simply changing some signs if necessary, c_n determines a cooling function c'_n for $(S, F_{n+1}, R_{n+1}, h_n)$ except for possibly taking the value -1 at some edges. The value of c'_n at every edge of F_{n+1} is strictly less than 1. As at the beginning of the passage from step n to

step $n + 1$, this implies that by appropriately increasing every value of c'_n by a positive amount we obtain a cooling function for $(S, F_{n+1}, R_{n+1}, h_n)$ which satisfies every conditional inequality for a real number strictly larger than h_n . Because the vertex inequality for c'_n is an equality for every vertex of F_{n+1} which is not a root, every value of every cooling function for $(S, F_{n+1}, R_{n+1}, h'_n)$ with $h'_n \geq h_n$ is at least as large as the corresponding value of c'_n . Maximizing, we obtain a real number h_{n+1} and a cooling function c_{n+1} for $(S, F_{n+1}, R_{n+1}, h_{n+1})$ with norm 1 which never takes the value -1 such that h_{n+1} is maximal with respect to the property that there exists a cooling function for $(S, F_{n+1}, R_{n+1}, h_{n+1})$ with norm 1. The quadruple $(F_{n+1}, R_{n+1}, h_{n+1}, c_{n+1})$ satisfies all required conditions. This completes the description of the algorithm.

In this paragraph we show that this algorithm ends with a solution after finitely many steps. We have seen that h_n^{-1} is a relative Følner ratio for every n . Hence the numerator of h_n is bounded by $|E(S)|$ and the denominator of h_n is bounded by $|S|$. There are only finitely many possibilities for h_n . Since the sequence h_0, h_1, h_2, \dots is strictly increasing, it follows that the algorithm ends after finitely many steps.

The proof of the following theorem is based on this modified simplex algorithm. By comparing the proof and the algorithm, one sees that the algorithm finds the subset S_0 of the theorem.

Theorem 4.1. *The set S contains a maximal Følner-optimal subset S_0 , that is, a subset S_0 with maximum Følner ratio, such that every Følner-optimal subset of S is contained in S_0 .*

Proof. After choosing directions for the edges of $E(S)$, the absolute cooling theorem implies that there exists a cooling function $c : E(S) \rightarrow \mathbb{R}$ with norm 1 rel h_0 , where h_0 is the inverse of the maximum Følner ratio of a subset of S . Without loss of generality we assume that if $e \in \partial E(S)$, then e is directed away from S and that $c(e) = 1$.

Let T be a Følner-optimal subset of S . So $h_0^{-1} = FR(T)$. We direct every edge of $\partial E(T)$ away from T . By definition c satisfies the vertex inequality,

$$\sum_{i(e)=v} c(e) - \sum_{t(e)=v} c(e) \geq h_0$$

for every $v \in S$, and so this holds for every $v \in T$. We combine these vertex inequalities, one for every element of T , and obtain

$$\sum_{e \in \partial E(T)} c(e) \geq |T|h_0 = |\partial E(T)|.$$

Since $|c(e)| \leq 1$ for every $e \in \partial E(T)$ and the number of these summands equals $|\partial E(T)|$, we can make two conclusions. One is that every vertex inequality for T is actually an equality. The other is that $c(e) = 1$ for every $e \in \partial E(T)$.

We use these two conclusions to construct the maximal subset of S with maximum Følner ratio. For this we define a (nonreflexive, nonsymmetric and nontransitive) relation on S as follows. Given $u, v \in S$, we have that $u \sim v$ if and only if there exists $e \in E(S)$ directed so that $t(e) = u$, $i(e) = v$ and $c(e) < 1$. (The condition $c(e) < 1$ is equivalent to $c(e) \neq 1$.) Here we view c as being defined not only for every edge $e \in E(S)$ with its given direction but also for the opposite edge \bar{e} , so that $c(\bar{e}) = -c(e)$.

Let X be the set of vertices of S whose associated vertex inequality is strict. Let \bar{X} be the closure of X under the above relation. Conclusion one from above implies that every subset of S with maximum Følner ratio is contained in the complement of X . Conclusion two then implies that every subset of S with maximum Følner ratio is contained in the complement of \bar{X} .

Now let S_0 be the complement of \bar{X} . To prove the theorem it suffices to prove that $FR(S_0) = h_0^{-1}$, the maximum Følner ratio of a subset of S . For this we first observe that every vertex inequality is actually an equality for every element of S_0 . Furthermore, if $e \in \partial E(S_0)$ is directed away from S_0 , then $c(e) = 1$. Combining the vertex equalities associated to S_0 as in the second paragraph of this proof, we find that $|\partial E(S_0)| = |S_0|h_0$. This implies that $FR(S_0) = h_0^{-1}$.

This proves Theorem 4.1. □

5. THE RELATIVE COOLING THEOREM

Our plan for the rest of the paper is to construct cooling functions by finding a Følner-optimal subset $S_0 \subset S$, building an optimal cooling function c_0 on S_0 , and then building cooling functions on larger and larger subsets of S , extending c_0 layer by layer. A cooling function on a smaller subset pumps heat into the layers not yet considered. Thus we need to know when the extension is possible. This problem is dealt with by means of the relative cooling theorem, which is a corollary to the cooling theorem of Section 2. The only twist in the argument is that we do not apply the theorem to a subgraph of Γ but rather to a slightly modified subgraph.

Setting: We assume Γ , G , S , $E(S)$, and $\partial E(S)$ given, with edges $e \in E(S)$ oriented as before.

The modified graph Γ'_S and modified vertex set S' : We form a new graph Γ'_S from the graph $\Gamma_S = \cup\{e \mid e \in E(S)\}$ by splitting all edges $e \in \partial E(S)$ apart at any common vertex in the complement of S so that the boundary edges have distinct terminal vertices in the complement of S . Let S' be a subset of the set of vertices of Γ'_S with $S \subset S'$.

Initial conditions: We assume given a function $h_0 : S' \rightarrow (0, \infty)$.

We assume: S' admits a cooling function $\text{rel } h_0$.

Lemma 2.1 implies that S' admits a cooling function $\text{rel } h_0$ if and only if every connected component of S' has a nonempty boundary.

We immediately obtain the following *relative cooling theorem*, which is simply the general cooling theorem of Section 2 applied to our modified graph Γ'_S :

Theorem 5.1 (Relative cooling). *(1) If N is the minimum possible norm of a cooling function $\text{rel } h_0$, then the reciprocal $1/N$ is given by the minimum value of a convex function $A(f)$ on the simplex $\Delta(S')$ as follows:*

$$1/N = \min_{f \in \Delta(S')} A(f),$$

where \tilde{f} is the modification of f defined above and

$$A(f) = \sum_{e \in E(S')} |\tilde{f}(i(e)) - \tilde{f}(t(e))|.$$

(2) The minimum possible norm N is given by the maximum Følner ratio:

$$N = \max_{S_0 \subset S'} FR(S_0).$$

(3) If $FR(S_0)$ realizes the maximum in (2), then the function $f \in \Delta(S')$ whose modification \tilde{f} is constant on S_0 and 0 on $S' \setminus S_0$ realizes the minimum in (1).

We shall apply this relative cooling theorem in Section 6.

6. PEELINGS

We assume Γ , G , S , $E(S)$, $\partial E(S)$, and $h_0 : S \rightarrow (0, \infty)$ given as before. Our goal is to find a Følner-optimal subset $S_0 \subset S$. We propose to do so by peeling layers away from S until we find the desired set.

Let P be a nonempty subset of S , and let $T = S \setminus P$. We have the (split) graph Γ'_P from the previous section. Let P' be the subset of Γ'_P consisting of P together with the boundary vertices of Γ'_P arising from (the splitting of) points of T . If $P_0 \subset P'$, then we let $B(P_0)$ denote the cardinality of $P_0 \setminus P$ and we let $B'(P_0)$ denote the number of edges in Γ'_P having exactly one vertex in P_0 . [Note that if $x \in P_0 \setminus P$ is not joined by its unique edge e in Γ'_P to a point of P_0 , then x is counted in $B(P_0)$ and e is counted in $B'(P_0)$.] We call P a *peeling* for S if

$$\forall P_0 \subset P', \quad H(P_0 \cap P) \leq FR(T)(B'(P_0) - B(P_0)).$$

The next lemma gives an alternate characterization of peelings.

Lemma 6.1. *Let N be a positive real number. In the above setting, we extend $h_0|_P : P \rightarrow (0, \infty)$ to P' by defining $h_0(x) = N$ for each $x \in P' \setminus P$. Then $FR(P_0) \leq N$ if and only if $H(P_0 \cap P) \leq N(B'(P_0) - B(P_0))$ for every $P_0 \subset P'$. In particular, taking $N = FR(T)$ shows that P is a peeling for S if and only if $FR(P_0) \leq FR(T)$ for every $P_0 \subset P'$.*

Proof. Let $P_0 \subset P'$. Then

$$FR(P_0) = \frac{N \cdot B(P_0) + H(P_0 \cap P)}{B'(P_0)}.$$

So $FR(P_0) \leq N$ if and only if

$$N \cdot B(P_0) + H(P_0 \cap P) \leq N \cdot B'(P_0)$$

if and only if

$$H(P_0 \cap P) \leq N(B'(P_0) - B(P_0)).$$

This proves Lemma 6.1. □

The next lemma gives a basic property of peelings.

Lemma 6.2. *If P is a peeling for S , then every connected component of $P' \subset \Gamma'_P$ has a nonempty boundary.*

Proof. Let P_0 be a nonempty subset of P' . From the definition of peeling,

$$H(P_0 \cap P) \leq FR(T)(B'(P_0) - B(P_0)).$$

If $B(P_0) = 0$, then $P_0 \subset P$, and so $H(P_0 \cap P) \neq 0$. It easily follows that $B'(P_0)$ is positive for every nonempty subset P_0 of P' . In particular, it is positive if P_0 is a connected component of P' . So some edge of Γ'_P contains exactly one vertex of P_0 . This proves Lemma 6.2. □

Peelings are important for two reasons:

Theorem 6.3. *If P is a peeling for $S = P \amalg T$, then every cooling function for T rel h_0 can be extended to a cooling function for S rel h_0 without increasing the cooling norm.*

Theorem 6.4. *If T is a proper subset of S which is Følner-optimal, then $S \setminus T$ is a peeling for S .*

These two theorems apply as follows: A subset S can be *peeled* until a subset S_0 is obtained that admits no peeling. When that happens, Theorem 6.4 implies that $FR(S_0)$ is as large as the Følner ratio of any of its subsets. Since S admits a cooling function rel h_0 , so does S_0 . Hence, by the cooling theorem, the set S_0 admits a cooling function c_0 of cooling norm $FR(S_0)$. By Theorem 6.3, this cooling function extends to a cooling function for S of norm $FR(S_0)$. We conclude that $FR(S_0)$ maximizes the Følner ratio of subsets of S , that $FR(S_0)$ is the minimal cooling norm for S , and that the extension of c_0 is an optimal cooling function for S . The following problems remain:

Problems. Give an efficient algorithm for determining when a peeling exists. Give an efficient algorithm for finding a peeling when a peeling exists.

Proof (6.3). Suppose we are given a cooling function for T rel h_0 . We can modify this function so that it is constant on all boundary edges of T and equal to the norm N of the function on those edges. It suffices to show that this modified cooling function extends to a cooling function for S rel h_0 , for returning the values of the extension on the boundary edges of T to their original values will not destroy the cooling properties of the extension.

From this point, we concentrate on $P' \subset \Gamma'_P$. We extend $h_0|_P : P \rightarrow (0, \infty)$ to P' by defining $h_0(x) = N$ for each $x \in P' \setminus P$. Lemmas 6.2 and 2.1 combine to show that P' admits a cooling function rel h_0 . Hence the relative cooling theorem applies to this situation. It implies that the smallest possible cooling norm for P' rel h_0 is $M = \max\{FR(P_0) \mid P_0 \subset P'\}$. Lemma 6.1 implies that $M \leq FR(T) \leq N$. Hence a cooling function for P' with norm M provides an extension of the original cooling function for T . This proves Theorem 6.3. \square

Proof (6.4). Let T be a proper subset of S whose Følner ratio $N = FR(T)$ is maximal among all subsets of S . Let $P = S \setminus T$, a nonempty subset of S . We prove that P is a peeling for S . Indeed, statement 2 of the cooling theorem implies that there exists a cooling function $c : E(S) \rightarrow \mathbb{R}$ for S rel h_0 of norm N . This function also cools T rel h_0 . With the edges of $\partial E(T)$

oriented away from T , the values of c on them must be constant with value N , for, if some boundary value carries less than the average N required of the boundary edges of T , then another would have to carry more than the average, a contradiction.

We return to Γ'_P as in the proof of Theorem 6.3. We again extend $h_0|_P : P \rightarrow (0, \infty)$ to P' by defining $h_0(x) = N$ for each $x \in P' \setminus P$. The edges of $E(P)$ are in canonical bijective correspondence with the edges of Γ'_P , and so c determines a function c' from the edges of Γ'_P to \mathbb{R} . It is clear that c' cools P viewed as a subset of Γ'_P , and c' also cools $P' \setminus P$ because every edge of $\partial E(T)$ carries N units of heat away from T . So c' is a cooling function relative to h_0 with norm N . Now we apply statement 2 of the relative cooling theorem to conclude that $FR(P_0) \leq N = FR(T)$ for every $P_0 \subset P'$. Now Lemma 6.1 implies that P is a peeling for S . \square

7. BUILDING OPTIMAL COOLING FUNCTIONS FOR $\mathbb{Z} \oplus \mathbb{Z}$

Our eventual goal (unfortunately, only partially completed in this paper) is to find an efficient algorithm to find a subset $S_0 \subset S$ having maximum Følner ratio $FR(S_0)$. The steps proposed are these:

- (1) By peeling, find a subset $T \subset S$ that allows no further peeling (or at least seems to allow no further peeling).
- (2) Prove that T is self-optimal ($FR(T) = \max\{FR(T_0) \mid T_0 \subset T\}$) by building a cooling function for T of norm $FR(T)$.
- (3) Extend the cooling function on T to a cooling function on S without increasing the cooling norm.

The relative cooling theorem, as applied in Theorem 6.3, completes step 3.

In the remainder of this section, we show how these steps can be carried out for the n -ball $B(n)$ in the free Abelian group $\mathbb{Z} \oplus \mathbb{Z}$ with its standard two-generator generating set. This ball is shaped like a diamond. Asymptotically, the Følner-optimal subset of $B(n)$ which we find is shaped like a regular octagon (stop sign). To guess the optimal shape, we simply look at successive Følner ratios as layers are removed at the corners: right, left, top, and bottom. Our first task is to determine when one of these edge layers is in fact a peeling. (Recall Section 6.) The theorems that follow will imply that, in these special cases, we need only compare a single fraction $m/2$ with the Følner ratio of the set before the deletion.

Setting: We consider the Cayley graph Γ of $G = \mathbb{Z}^2$ with standard generators and initial condition $h_0 \equiv 1$. We consider $T \subset \mathbb{Z}^2$ finite, T a subset of the lower open half-plane and containing $([1, m] \times \{-1\}) \cap \mathbb{Z}^2$ for some positive integer m . We consider $P = ([1, m] \times \{0\}) \cap \mathbb{Z}^2$. We ask whether P is a peeling for $P \cup T$. We return to $P' \subset \Gamma'_P$ as in Section 6.

Theorem 7.1. (1) If $P_0 \subset P' \setminus P$, then $H(P_0 \cap P) = 0$ and $B'(P_0) - B(P_0) = 0$.

(2) If $P_0 \subset P'$ and $P_0 \not\subset P' \setminus P$, then $B'(P_0) - B(P_0) > 0$ and the maximum

$$\frac{H(P_0 \cap P)}{B'(P_0) - B(P_0)}$$

is realized by the set $P_0 = P'$ with value $m/2$.

Corollary 7.2. The set P is a peeling for $S = T \cup P$ if and only if $m/2 \leq FR(T)$.

Remark. Of course, the search for a peeling P begins not with T but with $S = T \cup P$. We shall show later that this necessary inequality is equivalent to the inequality with $FR(S)$ replacing $FR(T)$ so that, as P changes, we need not calculate the Følner ratio of the new T but can instead retain the previously calculated old ratio $FR(S)$.

Proof (Theorem 7.1). Statement 1 and the inequality $B'(P_0) - B(P_0) > 0$ in statement 2 are easy to check. So let $P_0 \subset P'$ with $P_0 \not\subset P' \setminus P$. We alter P_0 by successive moves that can only increase the fraction being maximized.

There is a canonical injective graph morphism from Γ'_P to Γ , which we use to identify Γ'_P with a subgraph of Γ .

Without changing $B(P_0)$ and $H(P_0 \cap P)$, we may permute the elements of P in such a way that all elements of $P_0 \cap P$ appear consecutively beginning at $(1, 0)$. This can only decrease $B'(P_0)$ and thereby increase the fraction being maximized.

If $(a, 0) \in P_0$, then we may assume that $(a, -1) \in P_0$, for the insertion of $(a, -1)$ into P_0 will increase $B(P_0)$ by 1, will decrease $B'(P_0)$ by 1, and will leave $H(P_0 \cap P)$ unchanged.

If $(a, -1) \in P_0$ but $(a, 0) \notin P_0$, then we may assume such points appear just after the pairs described in the paragraphs above. Beginning with the left-most exemplar, we insert $(a, 0)$ into P_0 . This increases H by 1 and leaves $B(P_0)$ unchanged. Because $P_0 \not\subset P' \setminus P$, it also leaves $B'(P_0)$ unchanged.

If some $(a, 0) \in P$ does not appear in P_0 , then beginning with the leftmost such a , we insert both $(a, 0)$ and $(a, -1)$. This increases H by 1, $B(P_0)$ by 1, and $B'(P_0)$ by 1.

This easily proves Theorem 7.1. \square

We proceed to the improvement that allows us to substitute $FR(S)$ for $FR(T)$. Let n be a nonnegative integer, and let $k \in \{0, \dots, n\}$. We let $B(n, k)$ denote the set remaining after the first k horizontal layers are removed from $B(n)$ both from top and bottom and the first k vertical layers are removed from $B(n)$ from both the left and right. The set $B(n, k)$ is an octagon until k reaches $\lceil n/2 \rceil$, at which point $B(n, k)$ becomes a square.

It is helpful to allow the notation $* \in \{<, =, >\}$ for the next two theorems.

Theorem 7.3. *For each $* \in \{<, =, >\}$ and for each k with $0 \leq k < \lceil n/2 \rceil$,*

$$\begin{aligned} FR(B(n, k+1)) * FR(B(n, k)) &\quad \text{iff} \quad FR(B(n, k+1)) * (2k+1)/2 \\ &\quad \text{iff} \quad FR(B(n, k)) * (2k+1)/2. \end{aligned}$$

Proof. The set $B(n, k+1)$ is gotten from $B(n, k)$ by removing four layers from $B(n, k)$. Removing one layer removes $2k+1$ vertices and decreases the number of boundary edges by 2. So $B(n, k)$ has $x = 4(2k+1)$ more elements than $B(n, k+1)$ and $y = 8$ more boundary edges. Hence if $FR(B(n, k)) = \gamma_k/\delta_k$, then

$$\frac{\gamma_k}{\delta_k} = \frac{\gamma_{k+1} + x}{\delta_{k+1} + y} \quad \text{and} \quad \frac{\gamma_{k+1}}{\delta_{k+1}} = \frac{\gamma_k - x}{\delta_k - y}.$$

A bit of algebra now yields that

$$\frac{\gamma_{k+1}}{\delta_{k+1}} * \frac{\gamma_k}{\delta_k} \quad \text{iff} \quad \frac{\gamma_{k+1}}{\delta_{k+1}} * \frac{x}{y} \quad \text{iff} \quad \frac{\gamma_k}{\delta_k} * \frac{x}{y}.$$

This proves Theorem 7.3. \square

Theorem 7.4. *Let k_0 be the smallest integer greater than*

$$(2n - \sqrt{2n^2 + 2n + 1})/2.$$

Then $0 \leq k_0 \leq n/2$ and $FR(B(n, k))$ increases monotonically on the interval $0 \leq k \leq k_0$ and decreases monotonically on the interval $k_0 \leq k \leq \lceil n/2 \rceil$.

Proof. It is an easy matter to calculate $FR(B(n, k))$:

$$FR(B(n, k)) = \frac{2n^2 + 2n + 1 - 4k^2}{8n + 4 - 8k}.$$

Thus

$$FR(B(n, k)) * \frac{2k+1}{2}$$

if and only if

$$2n^2 + 2n + 1 - 4k^2 \quad * \quad 8nk + 4k - 8k^2 + 4n + 2 - 4k$$

if and only if

$$4k^2 + (-8n)k + (2n^2 - 2n - 1) \quad * \quad 0.$$

Set $p(x) = 4x^2 - 8nx + 2n^2 - 2n - 1$. The smaller root of $p(x)$ is

$$\begin{aligned} \frac{8n - \sqrt{64n^2 - 16(2n^2 - 2n - 1)}}{8} &= \frac{8n - \sqrt{32n^2 + 32n + 16}}{8} \\ &= \frac{2n - \sqrt{2n^2 + 2n + 1}}{2}. \end{aligned}$$

Since the larger root of $p(x)$ is positive and $p(-1) > 0$, it follows that $k_0 \geq 0$. One verifies that $p(\frac{n-1}{2}) < 0$. If n is odd, then this implies that $k_0 \leq (n-1)/2 < n/2$ and if n is even, then $k_0 \leq n/2$. Thus $0 \leq k_0 \leq n/2$. Moreover, from the last display we conclude that $FR(B(n, k)) > (2k+1)/2$ for $k < k_0$ and $FR(B(n, k)) < (2k+1)/2$ for $k_0 \leq k \leq \lceil n/2 \rceil$. (Note that since every term in $p(x)$ is even except the last, $p(x)$ has no integer roots.) Therefore the desired result follows from Theorem 7.3. \square

Theorem 7.5. *For every integer k with $0 \leq k < k_0$, the set $B(n, k) \setminus B(n, k+1)$ is a peeling for $B(n, k)$.*

Proof. Theorem 7.4 implies that $FR(B(n, k))$ increases monotonically on the interval $0 \leq k \leq k_0$. This and Theorem 7.3 imply that $FR(B(n, k+1)) \geq (2k+1)/2$ if $0 \leq k < k_0$. Let $P = B(n, k) \setminus B(n, k+1)$. Because $k_0 \leq n/2$, it follows that the graph Γ'_P is the disjoint union of four graphs, one for each of the four layers in P . Theorem 7.1 applies to each of these four layers, and we see that its conclusion holds even for P . As in Corollary 7.2, it follows that P is a peeling for $B(n, k)$ if and only if $m/2 \leq FR(B(n, k+1))$, where $m = 2k+1$. This completes the proof of Theorem 7.5. \square

Theorem 7.6. *The set $B(n, k_0)$ is asymptotically a regular octagon.*

Proof. The set is clearly an octagon with two horizontal sides of lengths approximately $2k_0$, two vertical sides of lengths approximately $2k_0$, and four diagonal sides at angles of forty-five degrees from the axes, each having (as yet unknown) approximate length ℓ . We want to show that $\ell \approx 2k_0$. Each diagonal side is the hypotenuse of an isosceles right triangle whose legs are approximately of length $n - 2k_0$. We calculate:

$$\frac{n}{k_0} \approx \frac{2n}{2n - \sqrt{2n^2 + 2n + 1}} = \frac{2}{2 - \sqrt{2 + 2/n + 1/n^2}} \approx \frac{2}{2 - \sqrt{2}}.$$

Thus,

$$\frac{\ell}{k_0} \approx \sqrt{2} \frac{(n - 2k_0)}{k_0} \approx \sqrt{2} \left(\frac{2}{2 - \sqrt{2}} - 2 \right) = 2.$$

We conclude that the octagon is almost regular for large values of n . \square

Theorem 7.7. *Let k_0 be the smallest integer greater than*

$$(2n - \sqrt{2n^2 + 2n + 1})/2.$$

Then $B(n, k_0)$ is a Følner-optimal subset of $B(n)$. That is, the Følner ratio of $B(n, k_0)$ is the maximal Følner ratio of subsets of $B(n)$. Every optimal cooling function on $B(n, k_0)$ can be extended to a cooling function on $B(n)$ without increasing the cooling norm.

Remark. *The following are the major steps in the proof.*

(1) *The set $B(n, \lceil n/2 \rceil)$ is a square and is self-optimal because every xy -rectangle in $\mathbb{Z} \oplus \mathbb{Z}$ is self-optimal.*

(2) *If $k_0 \leq k < \lceil n/2 \rceil$ and $B(n, k+1)$ is self-optimal, then $B(n, k)$ is also self-optimal.*

(3) *If $0 \leq k < k_0$, then $B(n, k) \setminus B(n, k+1)$ is a peeling for $B(n, k)$. Hence each cooling function on $B(n, k+1)$ can be extended to a cooling function on $B(n, k)$ without increasing the cooling norm.*

Steps (1) and (2) will show that $B(n, k_0)$ is self-optimal. Step (3) will then show that $B(n, k_0)$ is a Følner-optimal subset of $B(n)$.

Step (3) is Theorem 7.5. Step (1) is easy and will be carried out next. Step (2) requires a fair amount of further work which we shall carry out after Step (1).

For both Step (1) and Step (2) of the proof, we will manipulate what we call *difference diagrams*. For the first time in this paper we will make use of the fact that we are considering the Cayley graph $\Gamma = \Gamma(G, C)$ of an infinite group G with finite, torsion-free generating set C . If S is a finite subset of the set of vertices G of Γ , and if $f : E(S) \rightarrow \mathbb{R}$ is a cooling function, then we associate with f a *difference diagram* D as follows:

$$D : S \times C \rightarrow \mathbb{R}$$

$$D(s, c) = f((s, c, sc)) - f((sc^{-1}, c, s)).$$

That is, $D(s, c)$ gives the net heat loss at s along the orbit of G defined by the group generator c . The difference diagram is a refinement of the heat loss function at a vertex that we used in previous sections. We call D a diagram since its values can be recorded at the vertices of the graph Γ in a pictorial diagram. In order to make the relationship completely clear with the heat loss function $h(s)$ defined earlier, we note that

$$h(s) = \sum_{i(e)=s} f(e) - \sum_{t(e)=s} f(e) = \sum_{c \in C} D(s, c).$$

We need only this simple difference diagram in carrying out Step (1) of the proof. However, in Step (2) we shall consider the difference between difference diagrams defined on two different sets, one containing the other. In order to do this, we shall have to extend by 0 the difference diagram on the smaller set so that the two difference diagrams have the same domain of definition. This new difference will be called a *diagram of second differences*.

Theorem 7.8. *The cooling function f is determined by its difference diagram and the values of f on $\partial E(S)$.*

Corollary 7.9. *If S is self-optimal and $f : E(S) \rightarrow \mathbb{R}$ is a cooling function of norm $FR(S)$, then f is determined by its difference diagram.*

Proof (7.8 and 7.9). The corollary follows immediately from the theorem since, in the case of self-optimal sets S , the boundary values of optimal cooling functions are constant and equal to $FR(S)$.

In general, since every generator $c \in C$ has infinite order and the set S is finite, every c -orbit determines edges e_1, e_2, \dots, e_n , with the initial vertex $i(e_1)$ of e_1 and the terminal vertex $t(e_n)$ of e_n lying in the complement of S , all other vertices lying in S and $t(e_i) = i(e_i)c$ for $i \in \{1, \dots, n\}$. The edge e_1 is a boundary edge so that $f(e_1)$ is given. Inductively, $f(e_{i+1}) = f(e_i) + D(t(e_i), c)$. \square

Remark ((Important!)). *In the self-optimal case, it is important to realize that heat loss is always outwards at the boundary. This means that in the previous paragraph, the first value $f(e_1) = -FR(S)$ must be thought of as a negative number and the terminal value $f(e_n) = FR(S)$ as a positive value, with intermediate values $f(e_i)$ increasing, not necessarily monotonically, from the one value to the other. It is important that in its possibly oscillatory traverse from the negative value to the positive value, the values $f(e_i)$ never stray from the interval $[-FR(S), FR(S)]$. In terms of the difference diagram, this means that the partial sums along initial segments of the orbit remain in the interval $[0, 2 \cdot FR(S)]$, with the total sum equalling $2 \cdot FR(S)$.*

In the simplest cases, along each orbit the difference diagram exhibits a partition of $[0, 2 \cdot FR(S)]$; that is, the values of D are positive and sum to $2 \cdot FR(S)$. However, a partition does not always suffice. When finding a Følner-optimal subset of $B(n)$, we shall often have to employ an oscillatory traverse along certain orbits with D exhibiting both positive and negative values. The first case where this seems to be necessary is $B(8)$.

Theorem 7.10. *In $G = \mathbb{Z} \oplus \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$, every xy -rectangle R is self-optimal.*

Proof. Suppose R denotes the xy -rectangle $([1, m] \times [1, n]) \cap \mathbb{Z}^2$. Then

$$FR(R) = \frac{m \cdot n}{2(m+n)}.$$

We express our cooling function in terms of the unit $1/(2(m+n))$ so that our difference diagram can have integral entries. Boundary values are thus given (in terms of the prescribed units) by $\pm m \cdot n$ and the difference diagram must have initial partial sums that traverse from 0 to $2 \cdot m \cdot n$, never leaving the interval $[0, 2 \cdot m \cdot n]$. The negative sign is used near initial points of orbits, the positive sign near terminal points of orbits. A difference diagram is given by the function

$$D((x, y), a) = 2 \cdot n \quad \text{and} \quad D((x, y), b) = 2 \cdot m.$$

Since $D((x, y), a) + D((x, y), b) = 2 \cdot (m+n)$, each vertex is cooled by exactly $2(m+n)$ fractional units, hence by exactly 1 real unit. Along a -orbits and b -orbits, the difference diagram partitions $2 \cdot m \cdot n$. Hence, the cooling norm is $FR(R)$. \square

Theorem 7.11. *The set $B(n, k)$ is self-optimal for all integers k in the interval $[k_0, \lceil n/2 \rceil]$.*

Proof. Since the xy -square $B(n, \lceil n/2 \rceil)$ is self-optimal by the previous theorem, it is clearly enough to prove that, if $\lceil n/2 \rceil > k \geq k_0$ and $B(n, k+1)$ is self-optimal, then $B(n, k)$ is also self-optimal. Let k be an integer in the interval $[k_0, \lceil n/2 \rceil - 1]$.

We introduce $B(n, k+1, i)$, for $i \in \{0, 1, 2, 3, 4\}$, as the set formed from $B(n, k+1)$ by adding i of the layers of $B(n, k)$ not in $B(n, k+1)$. We let $FR(B(n, k+1, i)) = \alpha_i/\beta_i$. It suffices to show that, if $B(n, k+1, i)$ is self-optimal, then $B(n, k+1, i+1)$ is self-optimal, for $i \in \{0, 1, 2, 3\}$. We shall need the inequalities given in the following lemma.

Lemma 7.12. *If $j \in \{0, 1, 2, 3, 4\}$, then*

$$(2k+1)(\beta_j - 2(2k+1)) \leq 2\alpha_j < (2k+1)\beta_j.$$

Proof. To prove the second inequality, suppose that $FR(B(n, k)) = \gamma_k/\delta_k$, as in the proof of Theorem 7.3. Because $k_0 \leq k \leq \lceil \frac{n}{2} \rceil - 1$, Theorems 7.3 and 7.4 combine to imply that $\gamma_k/\delta_k < (2k+1)/2$. Hence

$$2\alpha_j = 2(\gamma_k + j(2k+1)) < (2k+1)(\delta_k + 2j) = (2k+1)\beta_j.$$

This proves the second inequality.

We prove the first inequality first for the case $j = 4$, where $B(n, k+1, 4) = B(n, k)$. The inequality is then equivalent to the inequality

$$(2k+1)(\beta_4 - 2(2k+1)) \leq 2\alpha_4.$$

Substituting known values for $\alpha_4 = |B(n, k)|$ and $\beta_4 = |\partial E(B(n, k))|$, as in the proof of Theorem 7.4, we seek to prove that

$$(2k + 1)(8n + 4 - 8k - 2(2k + 1)) \leq 2(2n^2 + 2n + 1 - 4k^2).$$

This reduces to

$$4k^2 + (2 - 4n)k + (n^2 - n) \geq 0,$$

equivalently,

$$(2k - n)(2k - (n - 1)) \geq 0.$$

Since $k \leq \lceil n/2 \rceil - 1 \leq (n - 1)/2$, the proof is complete for $j = 4$.

Now we induct downward for $j = 3, 2, 1, 0$. In the original inequality, before manipulation, each move downward subtracts exactly $2(2k + 1)$ from each side of the inequality. This completes the proof. \square

Assuming inductively that $B(n, k + 1, i)$ is self-optimal, we let D denote the difference diagram of some optimal cooling function for $B(n, k + 1, i)$. We employ $1/\beta_i$ as unit and assume inductively that we can choose the entries of D to be integers. We extend this difference diagram by 0 to the larger set $B(n, k + 1, i + 1) \times C$.

We seek a difference diagram D_+ , expressed in units $1/\beta_{i+1} = 1/(\beta_i + 2)$, that will exhibit $B(n, k + 1, i + 1)$ as self-optimal and ask ourselves the properties that must be satisfied by the second difference $E = D_+ - D$. Even though the two diagrams are expressed in different units, each will have integer entries, and we express E as the integer difference of those entries. In other words, we manipulate the number of heat loss *units* required and compare the number of units in the two diagrams.

We label the rows of $B = B(n, k + 1, i + 1)$ by symbols R_0, R_1, \dots from top to bottom and assume that R_0 is the new row with $2k + 1$ elements.

We label the columns of B that contain an element of R_0 by symbols C_1, \dots, C_{2k+1} from left to right and call these columns the *central columns*.

We label the other columns of B by symbols S_1, S_2, \dots from left to right and call these columns the *side columns*.

We view the elements of B as cells for holding heat loss tokens. Our task is to deposit heat loss tokens in these cells subject to the following conditions.

(1) Row R_0 is assigned $2 \cdot \alpha_{i+1}$ tokens. This new row R_0 had no heat loss at all assigned by the cooling function for $B(n, k + 1, i)$; hence the difference function E must account for the total difference $2 \cdot \alpha_{i+1}$ required along every

orbit. All other rows and columns are assigned only $2(2k+1)$ tokens. This requirement reflects the fact that the old rows and columns require exactly $2(\alpha_{i+1} - \alpha_i) = 2(2k+1)$ more heat loss units than assigned to that row or column by the cooling function for $B(n, k+1, i)$.

(2) After distribution, each cell of R_0 is to have β_{i+1} tokens. At these vertices of the new orbit R_0 , the total heat loss recorded by E must be β_{i+1} since there was no heat loss there with the cooling function for $B(n, k+1, i)$. All other cells are to have 2 tokens. This requirement reflects the fact that the old vertices require only $\beta_{i+1} - \beta_i = 2$ more heat loss units than assigned to them by the cooling function for $B(n, k+1, i)$.

Heat-loss token distribution in the new row R_0 : Dividing, we find

$$2 \cdot \alpha_{i+1} = (2k+1)q + r,$$

where the quotient q and the remainder r are integers and $0 \leq r < 2k+1$. That $q < \beta_{i+1}$ follows from the second inequality of Lemma 7.12 with $j = i+1$. That $\beta_{i+1} - q \leq 2(2k+1)$ follows from the first inequality of Lemma 7.12 with $j = i+1$.

Thus we may place q of the tokens assigned to the row R_0 into each cell of R_0 , with one extra in each of the first r cells. We may then place either $\beta_{i+1} - q$ (or $\beta_{i+1} - q - 1$, as appropriate) column tokens into the cells of R_0 so that the total number of tokens in each cell is β_{i+1} , as required.

Distribution of the heat-loss tokens associated with the side columns:

In side column S_j , place 2 of the tokens assigned to S_j into each of the bottom $2k+1$ cells of S_j .

The critical region: Almost always there will be cells of the side columns which still contain no heat loss tokens. These cells will form two triangular regions. These two triangular regions together with the portions of the central columns at the same height form what we call the *critical region*. See Figure 1.

Distribution in the central columns beneath the critical region:

Consider a row R_j that lies beneath the critical region. The cells in that row which have not yet been filled with two heat loss tokens are precisely those in the central columns. Place 2 of the heat loss tokens assigned to R_j into each of those cells. This will use up all of the $2(2k+1)$ heat loss tokens assigned to that row.

Distribution of heat-loss tokens in the critical region: Fill that portion of the triangular regions in row R_j with 2 tokens assigned to R_j in

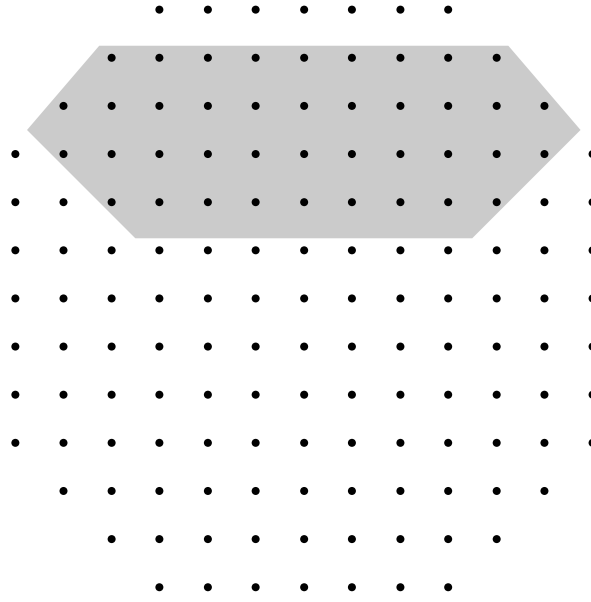


FIGURE 1. The set $B(9, 4, 4) = B(9, 3)$, whose critical region is within the shaded polygon.

each cell. For large n , the number of tokens needed for this distribution exceeds the number, $2(2k + 1)$, of tokens assigned to R_j . For each extra token needed, we create a new heat loss token and a compensating *negative* heat loss token. These negative heat loss tokens will be distributed in the intersection of the row R_j with the central columns.

In this paragraph we verify that the partial sums for E along initial segments of R_j stay in the range $[0, 2 \cdot (2k + 1)]$. It will then follow that the partial sums along initial segments of R_j for the associated cooling function stay in the range $[-FR(B(n, k + 1, i + 1)), FR(B(n, k + 1, i + 1))]$. The number of side columns left of the central columns which intersect R_j in the critical region is at most $n - 2k - 1$. In order for the partial sums for E along initial segments of R_j to be at most $2 \cdot (2k + 1)$, we need $n - 2k - 1 \leq 2k + 1$. This is equivalent to $k \geq (n - 2)/4$. As in the proof of Theorem 7.4, we let $p(x) = 4x^2 - 8nx + 2n^2 - 2n - 1$, and we verify that $p(n) < 0$ and

$$p\left(\frac{n-2}{4}\right) = \frac{(n-2)^2}{4} + 2n - 1 > 0.$$

Since k is between the smaller root of $p(x)$ and n , it follows that $k \geq (n - 2)/4$. This verifies that our partial sums are at most $2 \cdot (2k + 1)$. Symmetry shows that they are nonnegative. Thus the partial sums for E along initial segments of R_j stay in the range $[0, 2 \cdot (2k + 1)]$.

It remains only to distribute the remaining positive and negative row tokens and the remaining central column tokens to fill the central portion of the critical region.

We put all of the remaining column tokens in a single pot. We know exactly how many row tokens remain to be placed in each row, and in each row either every remaining token is positive or every remaining token is negative. This tells us exactly how many column tokens need to be placed in each row. We assign that number of column tokens to each row. Three paragraphs below we show that the pot contains exactly the correct number of tokens to do this.

We distribute the column tokens assigned to row R_1 as follows. We place the first column token assigned to row R_1 in column C_1 , the second in column C_2 , the third in column C_3 , and so on, continuing to column C_1 if necessary, until the column tokens assigned to R_1 have been exhausted.

Augment row and column numbers by 1 from where the last column token was inserted. Place the column tokens assigned to that row in successive columns. Iterate. When we are done, every column will contain the same number, $2k + 1$, of column tokens. Then we distribute the remaining row tokens so that the total value of the tokens in each of these cells is 2.

Now we show that the pot contains the correct number of tokens. Suppose that all of the tokens are distributed, possibly leaving some of these cells without the required two tokens. After distributing all of the tokens, the heat loss along every row and column is $2\alpha_{i+1}$ units. There are $\beta_{i+1}/2$ rows and columns, so the total number of units is $\alpha_{i+1}\beta_{i+1}$. Since there are α_{i+1} cells, the average number of units per cell is β_{i+1} . It follows that every cell has received the needed number of tokens. Similarly, if every cell has received the needed number of tokens, then the average heat loss along every row and column is $2\alpha_{i+1}$ units. This means that no tokens remain. The construction is then complete.

It remains only to note that the column distributions are partitions (no negative values for E). Although the row sums oscillate because of the negative tokens, again the norm stays in the desired bounds. Hence the associated cooling function has the correct norm.

Since we have constructed the required second difference E , we conclude that it is possible to construct a cooling function for $B(n, k + 1, i + 1)$ that has norm $FR(B(n, k + 1, i + 1))$, and the proof is complete. \square

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E-mail address: cannon@math.byu.edu

DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602, U.S.A.

E-mail address: floyd@math.vt.edu

DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG, VA 24061, U.S.A.
[HTTP://WWW.MATH.VT.EDU/PEOPLE/FLOYD](http://www.math.vt.edu/people/floyd)

E-mail address: walter.parry@emich.edu

DEPARTMENT OF MATHEMATICS, EASTERN MICHIGAN UNIVERSITY, YPSILANTI, MI 48197, U.S.A.